Method of Separation of Variables

1. Determine wether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.
You will be asked to solve only one of these six problems. 20 points

$$x\,u_{xx} + u_t = 0 \tag{1}$$

$$t\,u_{xx} + x\,u_t = 0 \tag{2}$$

$$u_{xx} + u_{xt} + u_t = 0 \tag{3}$$

$$[p(x)\,u_x]_x - r(x)\,u_{tt} = 0 \tag{4}$$

$$u_{xx} + u_{yy} + xu = 0 \tag{5}$$

$$u_{xx} + (x+y)u_{yy} = 0 \tag{6}$$

Follow the two examples 1D wave PDE and 1D Diffusion PDE (repeated below) we did in class.

$$u_{tt} = c^2\,u_{xx} \tag{7}$$

Solution: Let $u(x,t) = F(x)G(t)$, and substitute into the PDE.

$$F\ddot{G} = c^2 F''G$$

$$\frac{\ddot{G}}{c^2\,G} = \frac{F''}{F} = k$$

Yes, *separation of variables* can be used to replace $\boxed{u_{tt} = c^2\,u_{xx}}$ with a pair of ODEs. The resulting ODEs are

$$\begin{cases} F'' - kF &= 0 \\ \ddot{G} - kc^2G &= 0 \end{cases}$$

$$u_t = c^2\,u_{xx} \tag{8}$$

Solution: Let $u(x,t) = F(x)G(t)$, and substitute into the PDE.

$$F\dot{G} = c^2 F''G$$

$$\frac{\dot{G}}{c^2\,G} = \frac{F''}{F} = k$$

Yes, *separation of variables* can be used to replace $\boxed{u_t = c^2\,u_{xx}}$ with a pair of ODEs. The resulting ODEs are

$$\begin{cases} F'' - kF &= 0 \\ \dot{G} - kc^2G &= 0 \end{cases}$$

2. §12.4 D'Alembert's Solution of the Wave Equation 4 points
Show that because of the boundary conditions

$$\text{(a) } u(0,t) = 0, \qquad \text{(b) } u(L,t) = 0 \qquad \text{for all } t \geq 0$$

the function f in

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

must be odd and of period $2L$.