Gaussian Filters

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Linear Gaussian Systems

1. Kalman filter represents beliefs by the mean μ_t and the covariance Σ_t
2. Posteriors are Gaussian if
    1. State transition is "linear" with added Gaussian noise. (maybe)
    2. Measurement is also linear with added Gaussian noise.
    3. The initial belief $bel(x_0)$ is normally distributed.

State Transition Probability

1. State transition is "linear", with additive zero mean noise

$$x_t = A_t x_{t-1} + B_t u_t + \epsilon_t$$

2. Noise ϵ_t is zero mean, $\mathbb{E}\{\epsilon_t\} = 0$.
   And covariance $R_t = \mathbb{E}\{\epsilon_t \epsilon_t^T\}$
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$$\mathbb{E}\{(x_t - (A_t x_{t-1} + B_t u_t))(x_t - (A_t x_{t-1} + B_t u_t))^T\} = \mathbb{E}\{\epsilon_t \epsilon_t^T\} = R_t$$

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5. Hence the state transition probability is

$$p(x_t | u_t, x_{t-1}) = \frac{1}{\sqrt{|2\pi R_t|}} e^{-\frac{(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)}{2}}$$

Measurement Probability

1. Measurement is linear, with additive zero mean noise
$$z_t = C_t x_t + \delta_t$$
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5. Hence the measurement probability is
$$p(z_t|x_t) = \frac{1}{\sqrt{|2\pi Q_t|}} e^{-\frac{(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t)}{2}}$$

Initial belief $bel(x_0)$

1. The initial belief $bel(x_0)$ is normally distributed, with
   - mean μ_0
   - covariance Σ_0

$$bel(x_0) = p(x_0) = \frac{1}{\sqrt{|2\pi\Sigma_0|}} e^{-\frac{(x_0-\mu_0)^T \Sigma_0^{-1} (x_0-\mu_0)}{2}}$$

Kalman Filter Algorithm

1. KalmanFilter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$
2. $\overline{\mu}_t = A_t \mu_{t-1} + B_t\, u_t$
3. $\overline{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$
4. $K_t = \overline{\Sigma}_t C_t^T \left(C_t \overline{\Sigma}_t C_t^T + Q_t \right)^{-1}$
5. $\mu_t = \overline{\mu}_t + K_t \left(z_t - C_t \overline{\mu}_t \right)$
6. $\Sigma_t = \left(I - K_t C_t \right) \overline{\Sigma}_t$
7. return μ_t, Σ_t

Recall Bayes Filter Iteration

To begin with, we had:

- $bel(x_t) = p(x_t | z_{1:t}, u_{1:t})$
- $\overline{bel}(x_t) = p(x_t | z_{1:t-1}, u_{1:t})$

And Bayes Filter iterates the following:

$$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1} \tag{1}$$

$$bel(x_t) = \eta p(z_t | x_t)\, \overline{bel}(x_t) \tag{2}$$

Summary of Gaussian Linear Systems

$$x_t = A_t x_{t-1} + B_t u_t + \epsilon_t$$
$$z_t = C_t x_t + \delta_t$$
$$p(x_t|u_t, x_{t-1}) = \frac{1}{\sqrt{|2\pi R_t|}} e^{-\frac{(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)}{2}}$$
$$p(z_t|x_t) = \frac{1}{\sqrt{|2\pi Q_t|}} e^{-\frac{(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t)}{2}}$$
$$bel(x_0) = p(x_0) = \frac{1}{\sqrt{2\pi \Sigma_0}} e^{-\frac{(x_0 - \mu_0)^T \Sigma_0^{-1} (x_0 - \mu_0)}{2}}$$

and in general

$$bel(x_{t-1}) = p(x_{t-1}) = \frac{1}{\sqrt{2\pi \Sigma_{t-1}}} e^{-\frac{(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})}{2}}$$

Prediction. pages 45 to 53 from Probabilistic Robotics by Thrun, Burgard and Fox

$$\overline{bel}(x_t) = \int p(x_t|u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$$

$$= \int \frac{1}{\sqrt{|2\pi R_t|}} e^{-\frac{(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)}{2}}$$

$$\frac{1}{\sqrt{2\pi\Sigma_{t-1}}} e^{-\frac{(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})}{2}}\, dx_{t-1}$$

$$= \eta \int e^{-L_t}\, dx_{t-1}$$

here

$$L_t = \frac{(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)}{2} + \frac{(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})}{2}$$

Want to rewrite L_t as $\boxed{L_t = L_t(x_{t-1}, x_t) + L_t(x_t)}$ so that

$$\begin{aligned}\overline{bel}(x_t) &= \eta \int e^{-L_t}\, dx_{t-1} \\ &= \eta \int e^{-L_t(x_{t-1},x_t) - L_t(x_t)}\, dx_{t-1} \\ &= \eta e^{-L_t(x_t)} \int e^{-L_t(x_{t-1},x_t)}\, dx_{t-1}\end{aligned}$$

and finally

$$\overline{bel}(x_t) = \eta e^{-L_x(x_t)}$$

Derivatives of L_t

$$L_t = \frac{(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)}{2} + \frac{(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})}{2}$$

$$\frac{\partial L_t}{\partial x_{t-1}} = -A_t^T R_t^{-1} \left(x_t - A_t x_{t-1} - B_t u_t\right) + \Sigma_{t-1}^{-1} \left(x_{t-1} - \mu_{t-1}\right)$$

$$\frac{\partial^2 L_t}{\partial x_{t-1}^2} = A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1} =: \Psi_t^{-1} \qquad \text{for future use}$$

Set $\frac{\partial L_t}{\partial x_{t-1}} = 0$ and solve for x_{t-1}

$$\Sigma_{t-1}^{-1}\left(x_{t-1} - \mu_{t-1}\right) = A_t^T R_t^{-1}\left(x_t - A_t x_{t-1} - B_t u_t\right)$$

$$\Sigma_{t-1}^{-1} x_{t-1} - \Sigma_{t-1}^{-1}\mu_{t-1} = A_t^T R_t^{-1}\left(x_t - B_t u_t\right) - A_t^T R_t^{-1} A_t x_{t-1}$$

$$A_t^T R_t^{-1} A_t x_{t-1} + \Sigma_{t-1}^{-1} x_{t-1} = A_t^T R_t^{-1}\left(x_t - B_t u_t\right) + \Sigma_{t-1}^{-1}\mu_{t-1}$$

$$\underbrace{\left(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1}\right)}_{\Psi_t^{-1}} x_{t-1} = A_t^T R_t^{-1}\left(x_t - B_t u_t\right) + \Sigma_{t-1}^{-1}\mu_{t-1}$$

$$x_{t-1} = \Psi_t\left[A_t^T R_t^{-1}\left(x_t - B_t u_t\right) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]$$

Now we know one possible $L_t(x_{t-1}, x_t)$

$$L_t(x_{t-1}, x_t) = \frac{1}{2}\left(x_{t-1} - \Psi_t\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]\right)^T \Psi^{-1}$$
$$\left(x_{t-1} - \Psi_t\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]\right)$$

and since PDFs integrate to 1

$$\int \frac{1}{\sqrt{|2\pi\Psi|}} e^{-L_t(x_{t-1},x_t)}\, dx_{t-1} = 1$$
$$\int e^{-L_t(x_{t-1},x_t)}\, dx_{t-1} = \underbrace{\sqrt{|2\pi\Psi|}}_{\text{independent of } x_t}$$
$$\overline{bel}(x_t) = \eta e^{-L_t(x_t)} \int e^{-L_t(x_{t-1},x_t)} dx_{t-1}$$
$$= \eta e^{-L_t(x_t)}$$

Back to $L_t(x_t) = L_t - L_t(x_{t-1}, x_t)$

$$
\begin{aligned}
L_t(x_t) &= L_t - L_t(x_{t-1}, x_t) \\
&= \frac{(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)}{2} \\
&\quad + \frac{(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})}{2} \\
&\quad - \frac{1}{2}\left(x_{t-1} - \Psi_t \left[A_t^T R_t^{-1} (x_t - B_t u_t) + \Sigma_{t-1}^{-1} \mu_{t-1}\right]\right)^T \Psi^{-1} \\
&\qquad \left(x_{t-1} - \Psi_t \left[A_t^T R_t^{-1} (x_t - B_t u_t) + \Sigma_{t-1}^{-1} \mu_{t-1}\right]\right)
\end{aligned}
$$

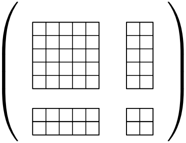
Next, expand and cancel terms with x_{t-1} (details omitted) to get

$$L_t(x_t) = +\frac{1}{2}(x_t - B_t u_t)^T R_t^{-1}(x_t - B_t u_t) + \frac{1}{2}\mu_{t-1}^T \Sigma_{t-1}^{-1}\mu_{t-1}$$
$$- \frac{1}{2}\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]^T \left(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1}\right)^{-1}$$
$$\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]$$

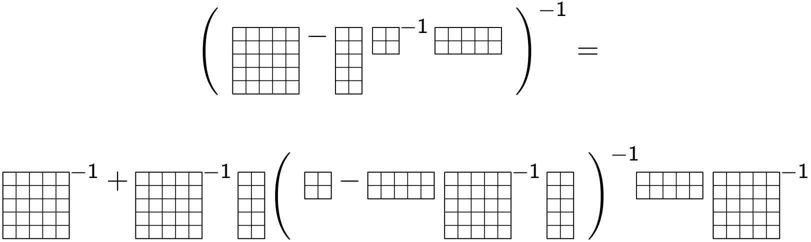
$$\frac{\partial L_t(x_t)}{\partial x_t} = R_t^{-1}(x_t - B_t u_t) - R_t^{-1} A_t \left(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1}\right)^{-1}$$
$$\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]$$
$$= \left[R_t^{-1} - R_t^{-1} A_t (A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} A_t^T R_t^{-1}\right](x_t - B_t u_t)$$
$$- R_t^{-1} A_t (A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} \Sigma_{t-1}^{-1}\mu_{t-1}$$

Matrix Inversion Lemma

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{c|c} A_{N\times N} & B_{N\times n} \\ \hline C_{n\times N} & D_{n\times n} \end{array}\right) =$$



$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}$$



Apply Matrix Inversion Lemma

$$\frac{\partial L_t(x_t)}{\partial x_t} = \underbrace{\left[R_t^{-1} - R_t^{-1}A_t(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} A_t^T R_t^{-1}\right]}_{(R_t + A_t \Sigma_{t-1} A_t^T)^{-1}} (x_t - B_t u_t)$$
$$- R_t^{-1}A_t(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} \Sigma_{t-1}^{-1} \mu_{t-1}$$
$$= (R_t + A_t \Sigma_{t-1} A_t^T)^{-1}(x_t - B_t u_t)$$
$$- R_t^{-1}A_t(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} \Sigma_{t-1}^{-1} \mu_{t-1}$$

Next, set $\frac{\partial L_t(x_t)}{\partial x_t} = 0$

$$(R_t + A_t \Sigma_{t-1} A_t^T)^{-1}(x_t - B_t u_t) = R_t^{-1}A_t(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} \Sigma_{t-1}^{-1} \mu_{t-1}$$

and solve for x_t.

Solving $\boxed{\frac{\partial L_t(x_t)}{\partial x_t} = 0}$ for x_t

$$x_t = B_t u_t + \underbrace{(R_t + A_t \Sigma_{t-1} A_t^T) R_t^{-1} A_t}_{A_t + A_t \Sigma_{t-1} A_t^T R_t^{-1} A_t} \underbrace{(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} \Sigma_{t-1}^{-1}}_{\left(\Sigma_{t-1} A_t^T R_t^{-1} A_t + I\right)^{-1}} \mu_{t-1}$$

$$= B_t u_t + A_t \underbrace{\left(I + \Sigma_{t-1} A_t^T R_t^{-1} A_t\right) \left(\Sigma_{t-1} A_t^T R_t^{-1} A_t + I\right)^{-1}}_{I} \mu_{t-1}$$

$$= B_t u_t + A_t \mu_{t-1} \qquad \text{this is line 2 of Kalman Algorithm}$$

Find $\frac{\partial^2 L_t(x_t)}{\partial x_t^2}$

$$\frac{\partial L_t(x_t)}{\partial x_t} = (R_t + A_t \Sigma_{t-1} A_t^T)^{-1}(x_t - B_t u_t) - R_t^{-1} A_t (A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} \Sigma_{t-1}^{-1} \mu_{t-1}$$

$$\frac{\partial^2 L_t(x_t)}{\partial x_t^2} = (R_t + A_t \Sigma_{t-1} A_t^T)^{-1}$$

this gives line 3 of Kalman Algorithm

Measurement Update

Need to compute

$$bel(x_t) = \eta p(z_t|x_t)\,\overline{bel}(x_t)$$

with

$$p(z_t|x_t) = \frac{1}{\sqrt{|2\pi Q_t|}} e^{-\frac{(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t)}{2}}$$

and

$$\overline{bel}(x_t) = \eta e^{-L_x(x_t)} = \eta e^{-\frac{(x_t - \overline{\mu}_t)^T \overline{\Sigma}_t^{-1} (x_t - \overline{\mu}_t)}{2}}$$

Hence

$$\begin{aligned} bel(x_t) &= \eta p(z_t|x_t)\,\overline{bel}(x_t) \\ &= \eta \frac{1}{\sqrt{|2\pi Q_t|}} e^{-\frac{(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)}{2}} e^{-\frac{(x_t - \overline{\mu}_t)^T \overline{\Sigma}_t^{-1}(x_t - \overline{\mu}_t)}{2}} \\ &= \eta e^{-J_t} \end{aligned}$$

here

$$J_t = \frac{(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)}{2} + \frac{(x_t - \overline{\mu}_t)^T \overline{\Sigma}_t^{-1}(x_t - \overline{\mu}_t)}{2}$$

Derivatives of J_t

$$J_t = \frac{(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t)}{2} + \frac{(x_t - \overline{\mu}_t)^T \overline{\Sigma}_t^{-1} (x_t - \overline{\mu}_t)}{2}$$

$$\frac{\partial J}{\partial x_t} = -C_t^T Q_t^{-1} (z_t - C_t x_t) + \overline{\Sigma}_t^{-1} (x_t - \overline{\mu}_t)$$

$$\frac{\partial^2 J}{\partial x_t^2} = C_t^T Q_t^{-1} C_t + \overline{\Sigma}_t^{-1}$$ this gives the covariance of $bel(x_t)$

$$\Sigma_t = \left(C_t^T Q_t^{-1} C_t + \overline{\Sigma}_t^{-1} \right)^{-1}$$

The mean of $bel(x_t)$ is the minimum of $\frac{\partial J}{\partial x_t}$.

$$\frac{\partial J}{\partial x_t} = -C_t^T Q_t^{-1}(z_t - C_t x_t) + \overline{\Sigma}_t^{-1}(x_t - \overline{\mu}_t) = 0$$

$$\overline{\Sigma}_t^{-1}(x_t - \overline{\mu}_t) = C_t^T Q_t^{-1}(z_t - C_t x_t) \qquad \text{set } x_t = \mu_t \text{ to get}$$

$$\overline{\Sigma}_t^{-1}(\mu_t - \overline{\mu}_t) = C_t^T Q_t^{-1}(z_t - C_t \mu_t)$$

$$\overline{\Sigma}_t^{-1}(\mu_t - \overline{\mu}_t) = C_t^T Q_t^{-1}(z_t - C_t \mu_t + C_t\overline{\mu}_t - C_t\overline{\mu}_t)$$

$$\overline{\Sigma}_t^{-1}(\mu_t - \overline{\mu}_t) = C_t^T Q_t^{-1}(z_t - C_t\overline{\mu}_t) - C_t^T Q_t^{-1} C_t(\mu_t - \overline{\mu}_t)$$

$$C_t^T Q_t^{-1}(z_t - C_t\overline{\mu}_t) = \underbrace{\left(C_t^T Q_t^{-1} C_t + \overline{\Sigma}_t^{-1}\right)}_{\Sigma_t^{-1}}(\mu_t - \overline{\mu}_t)$$

Kalman Gain $K_t = \Sigma_t C_t^T Q_t^{-1}$

The mean of $bel(x_t)$ is the minimum of $\frac{\partial J}{\partial x_t}$.

$$C_t^T Q_t^{-1}(z_t - C_t\overline{\mu}_t) = \underbrace{\left(C_t^T Q_t^{-1} C_t + \overline{\Sigma}_t^{-1}\right)}_{\Sigma_t^{-1}}(\mu_t - \overline{\mu}_t)$$

$$\underbrace{\Sigma_t C_t^T Q_t^{-1}}_{K_t}(z_t - C_t\overline{\mu}_t) = \mu_t - \overline{\mu}_t$$

$$\mu_t = \overline{\mu}_t + K_t(z_t - C_t\overline{\mu}_t) \quad \text{line 5}$$

Line 6: $\Sigma_t = (I - K_t C_t)\,\overline{\Sigma}_t$

$$\frac{\partial^2 J}{\partial x_t^2} = C_t^T Q_t^{-1} C_t + \overline{\Sigma}_t^{-1} \qquad \text{this gives the covariance of } bel(x_t)$$

$$\Sigma_t = \left(C_t^T Q_t^{-1} C_t + \overline{\Sigma}_t^{-1}\right)^{-1} \qquad \text{use Matrix Inversion Lemma}$$

$$= \overline{\Sigma}_t - \overline{\Sigma}_t C_t^T \left(Q_t + C_t \overline{\Sigma}_t C_t^T\right)^{-1} C_t \overline{\Sigma}_t$$

$$= \left[I - \underbrace{\overline{\Sigma}_t C_t^T \left(Q_t + C_t \overline{\Sigma}_t C_t^T\right)^{-1}}_{=K_t \quad \text{using 3.45 page 53}} C_t \right] \overline{\Sigma}_t$$

$$= (I - K_t C_t)\,\overline{\Sigma}_t \qquad \text{this is line 6 of Kalman Algorithm}$$