You must show **all** work to receive full credit. All work is to be your own. 04/04/2022

This is a closed books and notes test. Be organized. Total points: **102** 20:10-20:30

1. LMS update rule states

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \alpha e(n)\mathbf{x}(n)$$

Recall that deterministic gradient descent LMS analysis resulted in a step size bound

$$0 < \alpha < \frac{2}{\lambda_{\max}}$$

and since all eigenvalues of $\mathbf{R}_{N\times N}(n) = \mathbb{E}\{\mathbf{x}(n)\mathbf{x}^T(n)\}$ are nonnegative we have

$$0 < \alpha < \frac{2}{\sum_{i=1}^{N}\lambda_i} < \frac{2}{\lambda_{\max}}$$

(a) Show that $\mathrm{Tr}(\mathbf{R}) = \sum_{i=1}^{N}\lambda_i$. 5 points

Hint: $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$, and $\mathbf{R} = \mathbf{M\Lambda M}^T$, where $\mathbf{M}$ is orthogonal.

(b) Explain why $\mathrm{Tr}(\mathbf{R})$ corresponds to instantaneous tap input power, that is average energy of 5 points

$$\mathbf{x}(\mathtt{n:n-N+1,1}) = [x(n),\, x(x-1),\, \cdots,\, x(n-N+1)]^T$$

(c) Conclude from (a) and (b) the optimal LMS step size α using instantaneous estimates of tap input power. 3 points

(d) When computing crosscorrelations and autocorrelations we used time averages instead of ensemble averages, assuming that they are equal in our simulations. Describe the difference between the ensemble average $\mu(n) = \mathbb{E}\{\mathbf{x}(n)\}$ and the time average $\hat{\mu} = \frac{1}{M}\sum_{i=1}^{M} x(i)$.

8 points

(e) Comment on the use of step size

5 points

$$\alpha = \frac{1}{0.01 + \mathbf{x}(\texttt{n:n-N+1,1})^T \mathbf{x}(\texttt{n:n-N+1,1})}$$

instead of

$$\alpha = \frac{1}{\mathbf{x}(\texttt{n:n-N+1,1})^T \mathbf{x}(\texttt{n:n-N+1,1})}$$

2. System Identification and Tracking

Suppose RLS or LMS were used to track a linear system modeled as moving average with unknown constant coefficients $\mathbf{w} = [w_1\ w_2\ w_3\ w_4]^T$, also let the adaptive filter have four weights as well $\mathbf{w}_{\text{adpv}}$.

If the system excitation is given by

$$x(n) = \begin{cases} 1 & \text{if} \quad \mod(n,3) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the 4×4 autocorrelation matrix $\mathbf{R}_{4\times4}$ of $x(n)$. 10 points

(b) What is the rank of $\mathbf{R}_{4\times4}$? 5 points

(c) Will the adaptive filters converge to correct Wiener solution or not? Why? 5 points

3. Markov Processes; Stochastic Matrices   Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

   (a) Draw a weather (state) transition diagram describing this problem. 3 points

   (b) Write down a stochastic matrix $\mathbf{M}$ corresponding to the diagram above. 3 points

   (c) Write Matlab script to simulate weather (state) for N days. Your code accepts a positive integer N for number of days and generates a sequence of N weathers according to the Markov model specified above. For N around 10,000 the proportion of days for each weather *will* approach the eigenvector whose eigenvalue is 1. 10 points

   (d) Use Matlab's `[u v]=eig(M);` command to find the eigenvector $\mathbf{x}$ of $\mathbf{M}$ corresponding to *the* eigenvalue $\lambda = 1$, that is, the stationary distribution $\mathbf{x}$ that satisfies

$$\mathbf{x} = \mathbf{M}\mathbf{x}$$

   Normalize that eigenvector correctly (as a pdf, and not as Euclidean norm). Comment on, and compare this $\mathbf{x}$ with your result in (c). 4 points

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4. Stochastic 2-by-2 matrices

   Let $\mathbf{M} = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix}$ with $0 \le p \le 1$ and $0 \le q \le 1$.

   (a) Verify that $[1\ \ 1]$ is a *left eigenvector* of $\mathbf{M}$, that is $[1\ \ 1]\mathbf{M} = c[1\ \ 1]$. 3 points

   (b) What is the eigenvalue c corresponding to the *left eigenvector* $[1\ \ 1]$? 3 points

   (c) Use the facts $\boxed{\mathrm{Tr}(\mathbf{M}) = \lambda_1 + \lambda_2}$ and $\boxed{|\mathbf{M}| = \lambda_1 \cdot \lambda_2}$ to show that one of the eigenvalues $\{\lambda_1, \lambda_2\}$ equals to 1. 3 points

   (d) What is the other eigenvalue (the one not equal to 1)? 3 points

   (e) Find eigenvectors $\{\mathbf{v}_1\, \mathbf{v}_2\}$ corresponding to the eigenvalues $\{\lambda_1, \lambda_2\}$ respectively. 3 points

   (f) Verify correctness of your results in (e) by multiplying: $\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i$, $i \in \{1, 2\}$. 3 points

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5. Numerical example

   (a) Find the stable equilibrium distribution of the matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}$ 3 points

   (b) Verify the correctness of your result in (a). 3 points

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6. Stochastic n-by-n matrices
Recall that an $n \times n$ matrix $\mathbf{A}$ is said to be *stochastic* if the following conditions are satisfied

(a) Entries of $\mathbf{A}$ are non negative, that is $a_{i,j} \geq 0$ for all $1 \leq i \leq n$ and all $1 \leq j \leq n$.

(b) Each column of $\mathbf{A}$ sums to 1, that is $\sum_{i=1}^{n} a_{i,j} = 1$, for all $1 \leq j \leq n$.

Let $\mathbf{S}$ and $\mathbf{M}$ be arbitrary stochastic n-by-n matrices.

(a) Show that $\lambda = 1$ is an eigenvalue of $\mathbf{S}$. 3 points

(b) Show that $\mathbf{S}^2$ is also a stochastic matrix. 3 points

(c) Does $\mathbf{MS}$ have to be stochastic? Explain. 3 points