1. LMS update rule states

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \alpha e(n)\mathbf{x}(n)$$

Recall that deterministic gradient descent LMS analysis resulted in a step size bound

$$0 < \alpha < \frac{2}{\lambda_{\text{max}}}$$

and since all eigenvalues of $\mathbf{R}_{N\times N}(n) = \mathbb{E}\{\mathbf{x}(n)\mathbf{x}^T(n)\}$ are nonnegative we have

$$0 < \alpha < \frac{2}{\sum_{i=1}^{N}\lambda_i} < \frac{2}{\lambda_{\text{max}}}$$

(a) Show that $\text{Tr}(\mathbf{R}) = \sum_{i=1}^{N}\lambda_i$. 5 points

Hint: $\text{Tr}(\mathbf{AB}) = \text{Tr}(\mathbf{BA})$, and $\mathbf{R} = \mathbf{M\Lambda M}^T$, where $\mathbf{M}$ is orthogonal.

(b) Explain why $\text{Tr}(\mathbf{R})$ corresponds to instantaneous tap input power, that is average energy of 5 points

$$\mathbf{x}(\texttt{n:n-N+1,1}) = [x(n),\, x(x-1),\, \cdots,\, x(n-N+1)]^T$$

(c) Conclude from (a) and (b) the optimal LMS step size α using instantaneous estimates of tap input power. 3 points

(d) When computing crosscorrelations and autocorrelations we used time averages instead of ensemble averages, assuming that they are equal in our simulations. Describe the difference between the ensemble average $\mu(n) = \mathbb{E}\{\mathbf{x}(n)\}$ and the time average $\hat{\mu} = \frac{1}{M}\sum_{i=1}^{M} x(i)$.

8 points

(e) Comment on the use of step size

5 points

$$\alpha = \frac{1}{0.01 + \mathtt{x(n:n-N+1,1)}^T\mathtt{x(n:n-N+1,1)}}$$

instead of

$$\alpha = \frac{1}{\mathtt{x(n:n-N+1,1)}^T\mathtt{x(n:n-N+1,1)}}$$

2. System Identification and Tracking
Suppose RLS or LMS were used to track a linear system modeled as moving average with unknown constant coefficients $\mathbf{w} = [w_1\ w_2\ w_3]^T$, also let the adaptive filter have three weights as well $\mathbf{w}_{\text{adpv}}$. If the system excitation is given by

$$x(n) = \begin{cases} 1 & \text{if} \quad \bmod (n, 4) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the 3×3 autocorrelation matrix $\mathbf{R}_{3\times 3}$ of $x(n)$. 10 points

(b) What is the rank of $\mathbf{R}_{3\times 3}$? 5 points

(c) Will the adaptive filters converge to correct Wiener solution or not? Why? 5 points

3. Markov Processes; Stochastic Matrices On any given day Henry is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5.

   (a) Draw a mood (state) transition diagram describing this problem. 3 points

   (b) Write down a stochastic matrix $\mathbf{M}$ corresponding to the diagram above. 3 points

   (c) Write Matlab script to simulate Henry's moods for N days. Your code accepts a positive integer N for number of days and generates a sequence of N moods according to the Markov model specified above. For N around 10,000 the proportion of days for each mood *will* approach the eigenvector whose eigenvalue is 1. 10 points

   (d) Use Matlab's `[u v]=eig(M);` command to find the eigenvector $\mathbf{x}$ of $\mathbf{M}$ corresponding to *the* eigenvalue $\lambda = 1$, that is, the stationary distribution $\mathbf{x}$ that satisfies

$$\mathbf{x} = \mathbf{M}\mathbf{x}$$

   Normalize that eigenvector correctly (as a pdf, and not as Euclidean norm). Comment on, and compare this $\mathbf{x}$ with your result in (c). 4 points

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4. Stochastic 2-by-2 matrices

   Let $\mathbf{M} = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix}$ with $0 \le p \le 1$ and $0 \le q \le 1$.

   (a) Verify that $[1\ \ 1]$ is a *left eigenvector* of $\mathbf{M}$, that is $[1\ \ 1]\mathbf{M} = c[1\ \ 1]$. 3 points

   (b) What is the eigenvalue c corresponding to the *left eigenvector* $[1\ \ 1]$? 3 points

   (c) Use the facts $\boxed{\mathrm{Tr}(\mathbf{M}) = \lambda_1 + \lambda_2}$ and $\boxed{|\mathbf{M}| = \lambda_1 \cdot \lambda_2}$ to show that one of the eigenvalues $\{\lambda_1, \lambda_2\}$ equals to 1. 3 points

   (d) What is the other eigenvalue (the one not equal to 1)? 3 points

   (e) Find eigenvectors $\{\mathbf{v}_1\, \mathbf{v}_2\}$ corresponding to the eigenvalues $\{\lambda_1, \lambda_2\}$ respectively. 3 points

   (f) Verify correctness of your results in (e) by multiplying: $\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i$, $i \in \{1, 2\}$. 3 points

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5. Numerical example

   (a) Find the stable equilibrium distribution of the matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}$ 3 points

   (b) Verify the correctness of your result in (a). 3 points

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6. Stochastic n-by-n matrices

Recall that an $n \times n$ matrix $\mathbf{A}$ is said to be *stochastic* if the following conditions are satisfied

(a) Entries of $\mathbf{A}$ are non negative, that is $a_{i,j} \geq 0$ for all $1 \leq i \leq n$ and all $1 \leq j \leq n$.

(b) Each column of $\mathbf{A}$ sums to 1, that is $\sum_{i=1}^{n} a_{i,j} = 1$, for all $1 \leq j \leq n$.

Let $\mathbf{S}$ and $\mathbf{M}$ be arbitrary stochastic n-by-n matrices.

(a) Show that $\lambda = 1$ is an eigenvalue of $\mathbf{S}$. 3 points

(b) Show that $\mathbf{S}^2$ is also a stochastic matrix. 3 points

(c) Does $\mathbf{MS}$ have to be stochastic? Explain. 3 points