Matrix Inversion Lemma aka Woodbury Matrix Identity

Want to establish[1]

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}$$

known as Woodbury identity, and derive few of its consequences.

Let a square $(N+n)$-by-$(N+n)$ matrix be given

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{c|c} A_{N\times N} & B_{N\times n} \\ \hline C_{n\times N} & D_{n\times n} \end{array}\right) = \left(\begin{array}{ccccc|cccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} & b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} & b_{21} & b_{22} & \cdots & b_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3N} & b_{31} & b_{32} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} & b_{N1} & b_{N2} & \cdots & b_{Nn} \\ \hline c_{11} & c_{12} & c_{13} & \cdots & c_{1N} & d_{11} & d_{12} & \cdots & d_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2N} & d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nN} & d_{n1} & d_{n2} & \cdots & d_{nn} \end{array}\right)$$

To begin, consider *four* Block Gaussian Eliminations

$$\begin{pmatrix} I_N & W_{N\times n} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A + WC & \overbrace{B + WD}^{W=-BD^{-1}} \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} I_N & 0 \\ X_{n\times N} & I_n \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ \underbrace{XA + C}_{X=-CA^{-1}} & XB + D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_N & Y_{N\times n} \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A & \overbrace{AY + B}^{Y=-A^{-1}B} \\ C & CY + D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_N & 0 \\ Z_{n\times N} & I_n \end{pmatrix} = \begin{pmatrix} A + BZ & B \\ \underbrace{C + DZ}_{Z=-D^{-1}C} & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix} \tag{4}$$

[1] via a constructive derivation and not a mere verification.

Next, use Block Diagonalizations to obtain LDU factorization.

1. Applying (1) and (4) (they both involve D^{-1}), gives

$$\begin{pmatrix} I_N & -BD^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_N & 0 \\ -D^{-1}C & I_n \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}$$

$$\left[\begin{pmatrix} I_N & -BD^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_N & 0 \\ -D^{-1}C & I_n \end{pmatrix}\right]^{-1} = \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}^{-1}$$

Distribute the inverse, then multiply by appropriate factors on the right side to get

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I_N & 0 \\ -D^{-1}C & I_n \end{pmatrix}\begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}^{-1}\begin{pmatrix} I_N & -BD^{-1} \\ 0 & I_n \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} I_N & 0 \\ -D^{-1}C & I_n \end{pmatrix}\begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}\begin{pmatrix} I_N & -BD^{-1} \\ 0 & I_n \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} + D^{-1} \end{pmatrix} \tag{7}$$

2. Applying (2) and (3) (they both involve A^{-1}), gives

$$\begin{pmatrix} I_N & 0 \\ -CA^{-1} & I_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_N & -A^{-1}B \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}$$

$$\left[\begin{pmatrix} I_N & 0 \\ -CA^{-1} & I_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_N & -A^{-1}B \\ 0 & I_n \end{pmatrix}\right]^{-1} = \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}^{-1}$$

Distribute the inverse, then multiply by appropriate factors on the right side to get

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I_N & -AB^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}^{-1}\begin{pmatrix} I_N & 0 \\ -CA^{-1} & I_n \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} I_N & -AB^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix}\begin{pmatrix} I_N & 0 \\ -CA^{-1} & I_n \end{pmatrix} \tag{9}$$

$$= \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \tag{10}$$

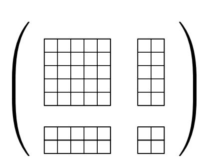
3. Compare $(1,1)$ entries in (7) and (10) to get

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} \tag{11}$$

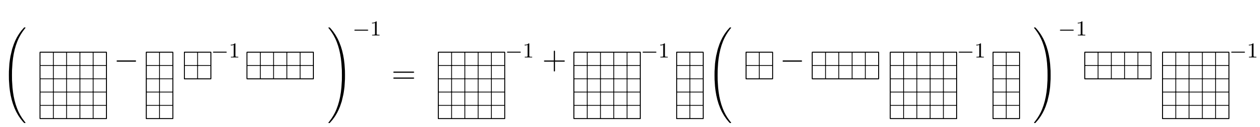
This is the Woodbury identity.

4. We can visualize the dimensions of the blocks of the Woodbury identity as follows

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{c|c} A_{N\times N} & B_{N\times n} \\ \hline C_{n\times N} & D_{n\times n} \end{array}\right) =$$



Then (11) looks like



5. As a special case consider the following. Let a square $(N+1)$-by-$(N+1)$ matrix be given

$$\begin{pmatrix} A & u \\ v^T & d \end{pmatrix} = \left(\begin{array}{c|c} A_{N\times N} & u_{N\times 1} \\ \hline v^T_{1\times N} & d_{1\times 1} \end{array}\right) = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} & u_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} & u_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3N} & u_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} & u_N \\ \hline v_1 & v_2 & v_3 & \cdots & v_N & d \end{array}\right)$$

And apply (11) (reproduced here for convenience)

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} \tag{11}$$

with $B = u$, $C = v^T$, and $D = d$, to get

$$\left(A - \frac{uv^T}{d}\right)^{-1} = A^{-1} + A^{-1}u\left(\underbrace{d - v^TA^{-1}u}_{1\times 1}\right)^{-1} v^TA^{-1} = A^{-1} + \frac{\overbrace{(A^{-1}u)(v^TA^{-1})}^{\text{rank 1}}}{d - v^TA^{-1}u} \tag{12}$$

Set $d = -1$ to get rank 1 inverse update formula

$$\left(A + uv^T\right)^{-1} = A^{-1} - \frac{(A^{-1}u)(v^TA^{-1})}{1 + v^TA^{-1}u} \tag{13}$$

Known as Sherman-Morrison formula.
Set $A = I$ to get

$$\left(I + uv^T\right)^{-1} = I - \frac{uv^T}{1 + v^Tu} \tag{14}$$

In (11), set $A = I_N$ and $D = I_n$ to get

$$(I_N + BC)^{-1} = I_N + B(I_n + CB)^{-1}C \tag{15}$$

Finally, in (11) set $N = n$, $B = C = I_N$, and $S = -D^{-1}$, then we get

$$\begin{aligned} (A+S)^{-1} &= A^{-1} - A^{-1}\underbrace{(S^{-1} + A^{-1})^{-1}A^{-1}}_{Q^{-1}R^{-1}=(RQ)^{-1}} \\ &= A^{-1} - A^{-1}\left[A(S^{-1} + A^{-1})\right]^{-1} \\ &= A^{-1} - \underbrace{A^{-1}\left[AS^{-1} + I\right]^{-1}}_{Q^{-1}R^{-1}=(RQ)^{-1}} \\ &= A^{-1} - \left[(AS^{-1} + I)A\right]^{-1} \\ &= A^{-1} - \left(AS^{-1}A + A\right)^{-1} \end{aligned}$$

In conclusion

$$(A+S)^{-1} = A^{-1} - \left(A + AS^{-1}A\right)^{-1} \tag{16}$$

Known as Hua's identity.