University of Detroit Mercy Mathematics Competition

February 27, 2018

1 Introduction

Definition 1. *The* arrangement graph $A_{n,k}$ *(with $n > k \geq 1$) has a vertex set consisting of all possible permutations of k elements chosen from the ground set of n elements $\{1, 2, \ldots, n\}$. Two vertices (nodes) u and v of $A_{n,k}$ are adjacent if their corresponding permutations differ in exactly one of the k positions.*

For example, the figure below shows the graph $A_{4,2}$. Here, $n = 4$, and $k = 2$, hence each vertex corresponds to a permutation on two symbols chosen from a set of four symbols $\{1, 2, 3, 4\}$. The vertex 34 has two neighbors 14 and 24, since they differ in the first position, and another two neighbors 31 and 32 since they differ in the second position.

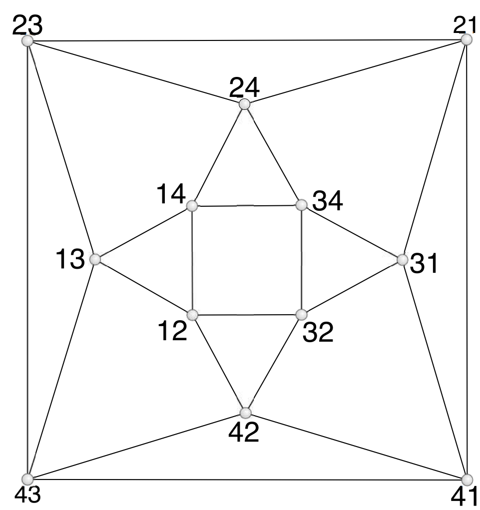

Figure 1: The graph $A_{4,2}$.

Definition 2. *A* hamiltonian cycle *in a graph is a closed path through a graph that visits each node exactly once.*

2 The Problem

Find 3 *edge disjoint* hamiltonian cycles (aka hamiltonian factors) for the graph $A_{5,2}$.

3 An Example

The graph $A_{4,2}$ has two edge disjoint hamiltonian cycles as shown in the figure below. The first hamiltonian cycle is drawn with bold edges, while the second is drawn with thin edges. Removing all the edges of the first hamiltonian cycle leaves the second hamiltonian cycle intact.

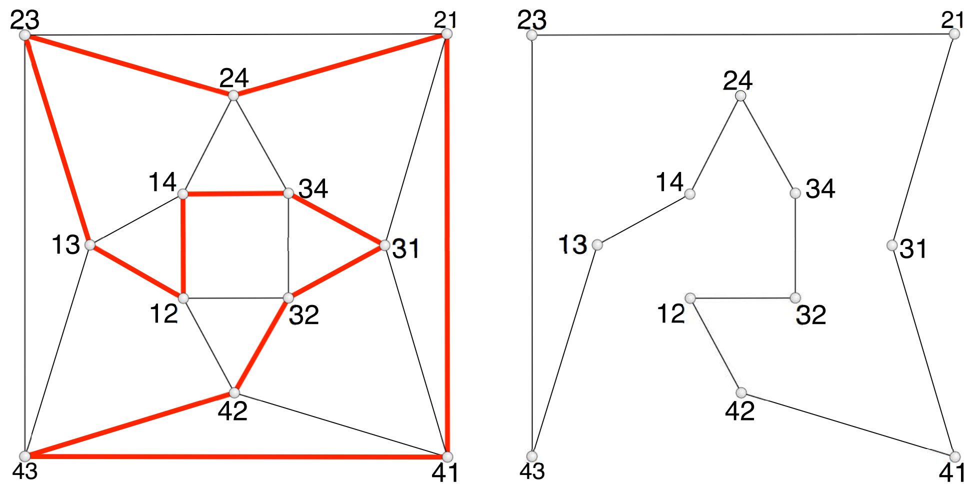

Figure 2: Two edge disjoint hamiltonian cycles in $A_{4,2}$

Be careful when finding hamiltonian cycles. The following figure shows yet another hamiltonian cycle in $A_{4,2}$, however the removal of its edges leaves two 3-cycles and one 6-cycle.

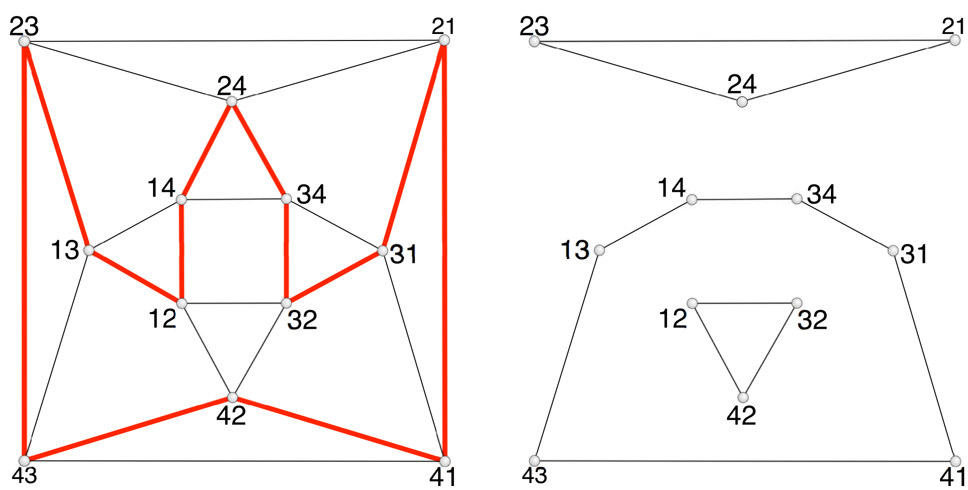

Figure 3: Four edge disjoint cycles in $A_{4,2}$. One hamiltonian, one of length 6, and two of length 3.

The the graph $A_{5,2}$ can not be drawn in a plane without crossing edges. Figures 4 and 5 show possible drawings of $A_{5,2}$.

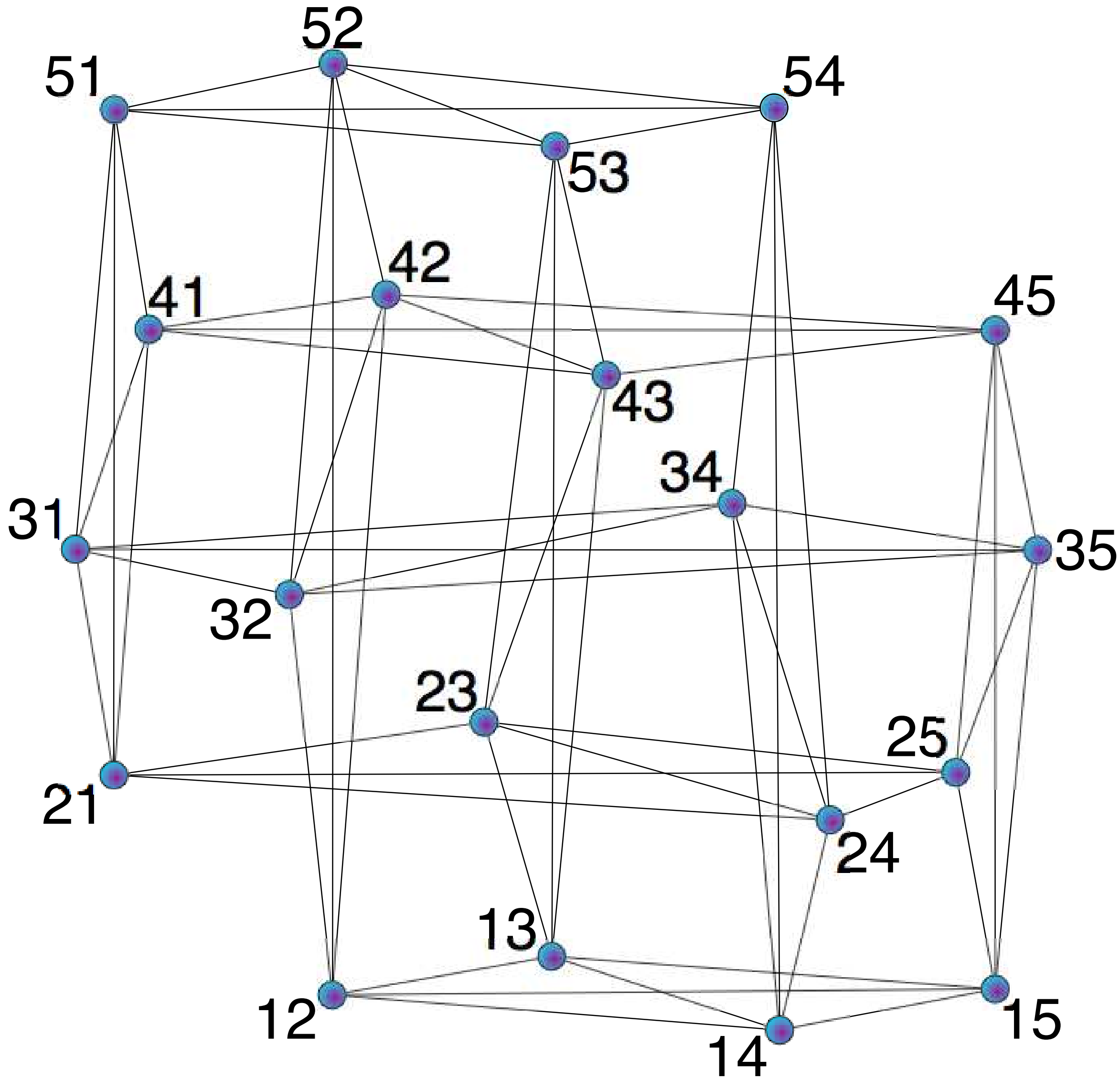

Figure 4: One possible drawing of $A_{5,2}$.

4 Extensions

Was your method of finding hamiltonian factors of $A_{5,2}$ systematic? That is, can you apply your approach to find *five* hamiltonian factors of $A_{7,2}$? How about 3 factors of $A_{5,3}$, or $A_{n,2}$ in general?

5 Open Problem(s)

- Find a hamiltonian factorization of the graph $A_{n,k}$ for even $k(n-k)$.
- Very important special case; find a hamiltonian factorization of $A_{n,n-1}$ for *odd* n.

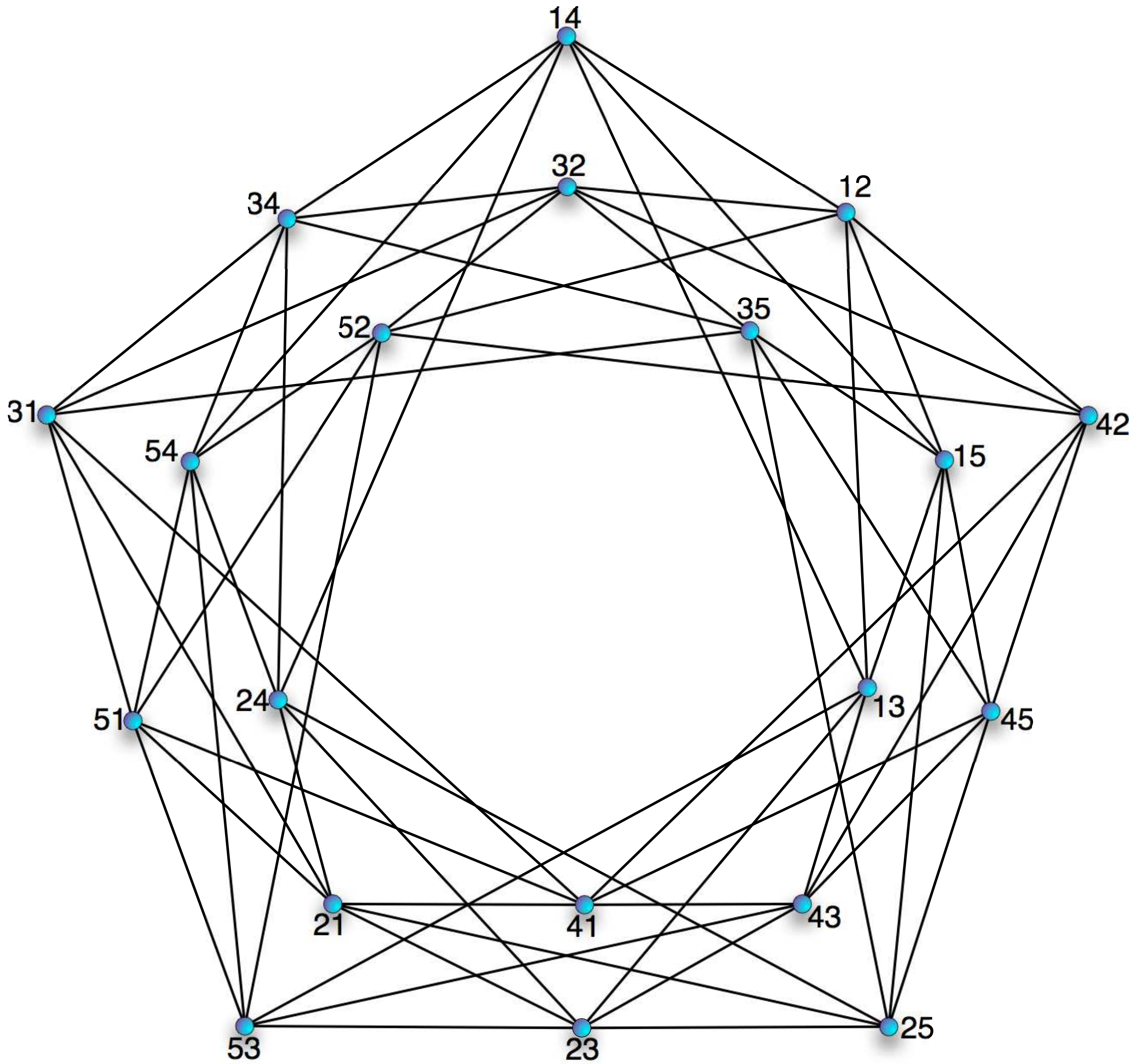

Figure 5: One more drawing of $A_{5,2}$.

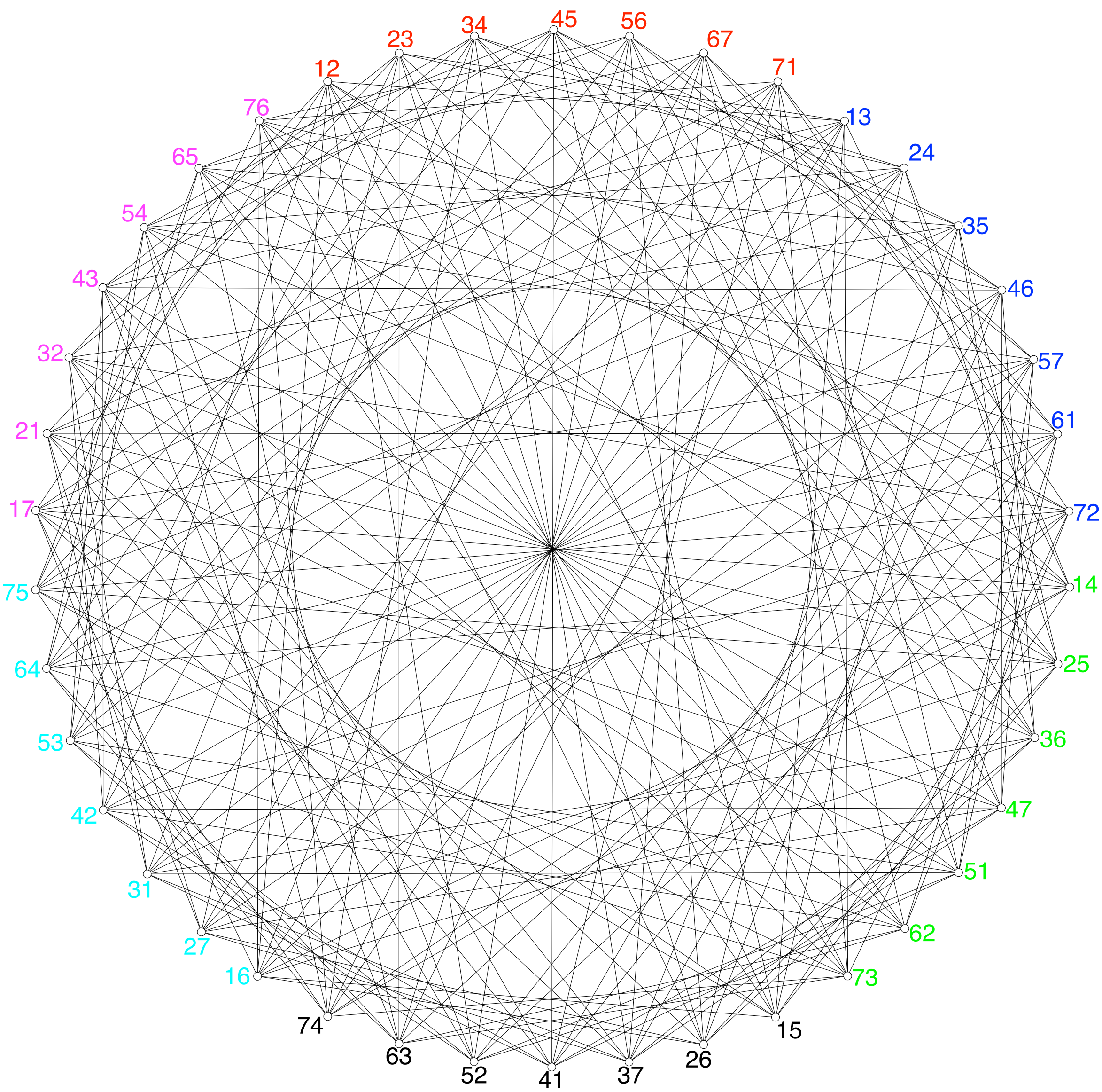

Figure 6: A drawing of $A_{7,2}$.